Floquet instabilities on Rossby Waves

- Context
  - 2D shallow water
  - Earth scale
- Hypotheses
  - β -plane : $f \approx f_0 + \beta y$
  - Rossby number:

$$Ro \triangleq \frac{U}{fL} << 1$$

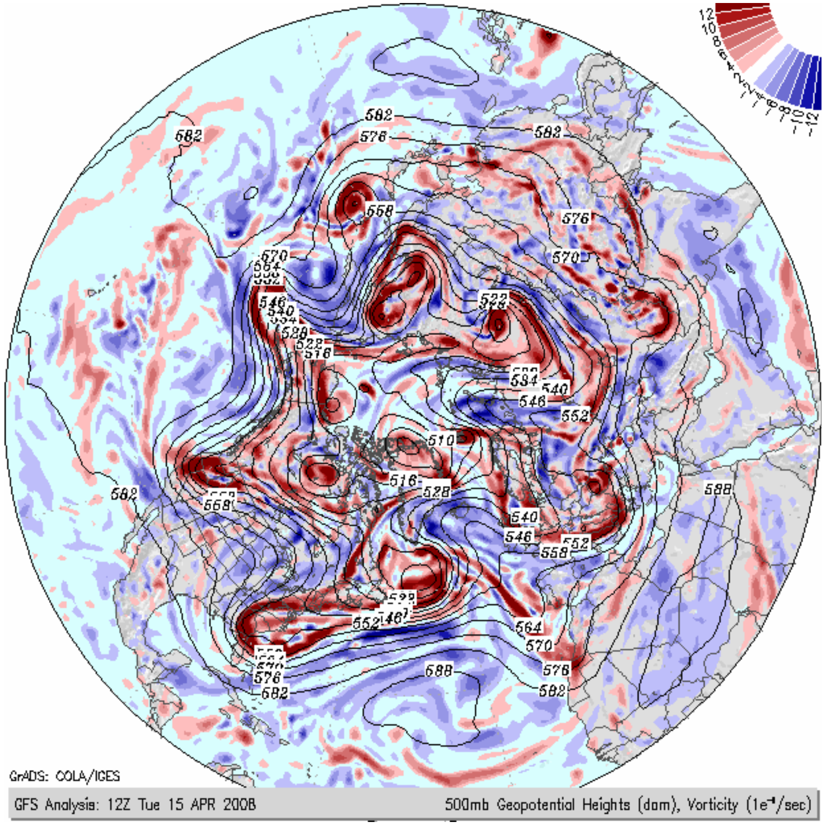

Fig. 1: How can one explain this strange belt around the North Pole?

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Equations of quasi-geostrophy

- Shallow water dimensional equations

$$\boxed{\begin{aligned} \frac{Du}{Dt} - (f_0 + \beta y)v &= -g'\frac{\partial h}{\partial x} \\ \frac{Dv}{Dt} + (f_0 + \beta y)u &= -g'\frac{\partial h}{\partial y} \\ \frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 \end{aligned}}$$

h : height of the fluid above mean topography

- Absolute vorticity approach
  - $\zeta_{abs} \triangleq \zeta + f = (v_x - u_y) + (f_0 + \beta y)$
  - Advected quantity: $\frac{D}{Dt}\left\{\frac{\zeta_{abs}}{h}\right\} = 0$

Potential vorticity

- Nondimensionalization
  - PV: $Q = \dfrac{1 + Ro.\beta y + Ro.\zeta}{1 + Ro.\Psi} \triangleq \underbrace{(1 + Ro.\beta y)}_{\text{leading}} + \underbrace{Ro.q}_{\text{correction}}$
  - Streamfunction Ψ such that $(u, v) = (-\Psi_y, \Psi_x)$
  - $\mathcal{O}(Ro)$ correction yields

$$q = \nabla^2 \Psi - \Psi \approx \nabla^2 \Psi$$ (barotropic assumption)

  - Barotropic vorticity equation

$$\boxed{\frac{DQ}{Dt} = 0 \Leftrightarrow \frac{D}{Dt}\left\{\nabla^2 \Psi + \beta y\right\} = 0}$$

Basic solution

- Form of the basic solution
  - Barotropic PDE: $\nabla^2\Psi_t + J(\Psi, \nabla^2\Psi) + \beta\Psi_x = 0$
  - Plane wave representation: $\Psi = \Psi(kx + ly - \omega t)$
- Basic solution with mean flow $\bar{U}$
  - Solution:

$$\Psi^B = U\kappa^{-1}\sin(kx + ly - \omega t) - \bar{U}y$$

  - Dispersion relation:

$$\omega = \bar{U}k - \beta k/\kappa^2$$

  - Assume stationarity:

$$\omega = 0$$

Undisturbed system

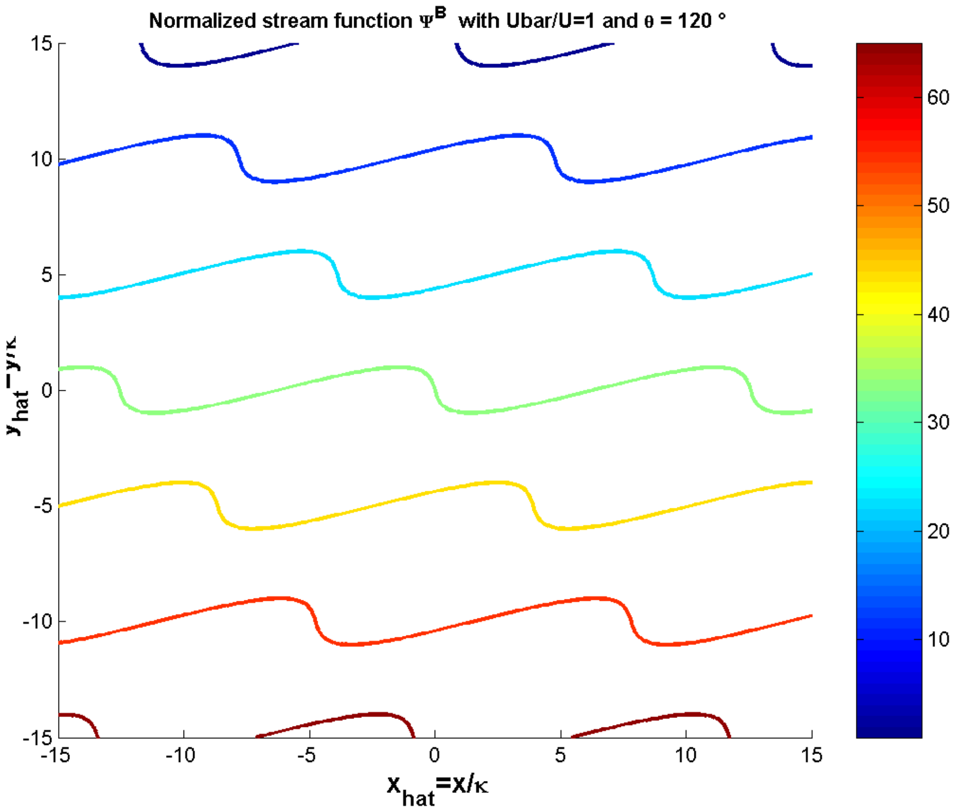

Fig. 2: Matlab *simulation of the basic streamfunction with wave vector oriented 120° w.r.t the x-axis*

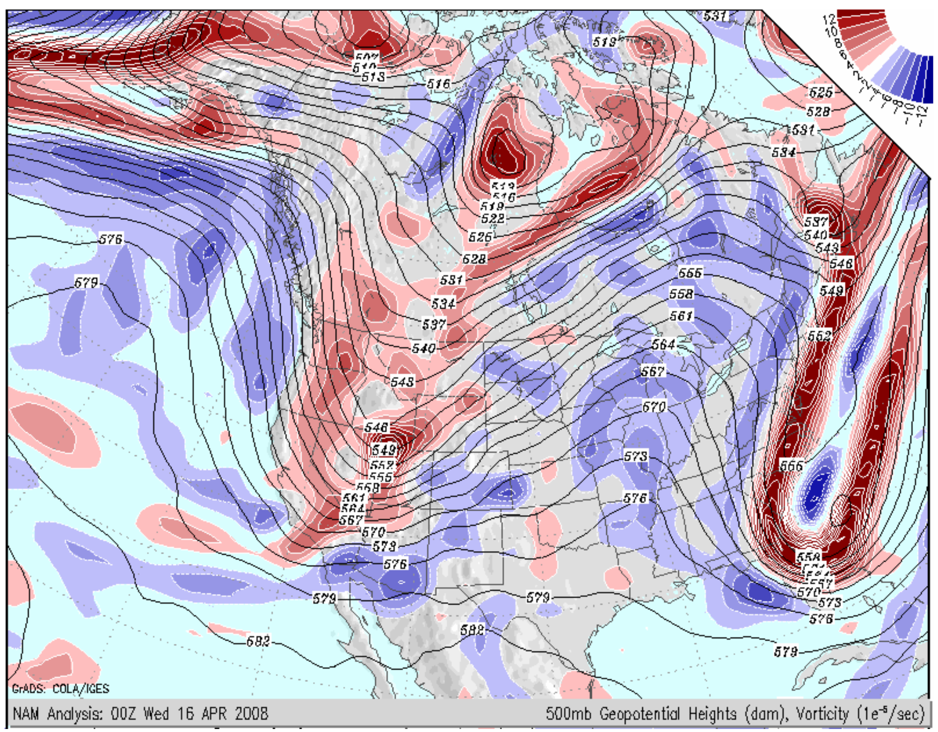

Fig. 3: Geopotential height and vorticity, Wed 16 Apr, 2008

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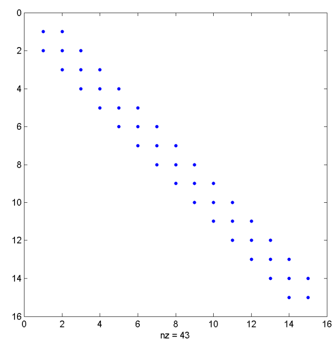
Perturbation streamfunction

- Disturbance: $\Psi = \Psi^B + \psi$
  - ➔ PDE for ψ
- Floquet theorem
  - Existence of solutions of the form
    $$\begin{aligned} \psi &= e^{i\lambda t} \times \text{periodic function in } x \text{ and } y \\ &= e^{i\lambda t} \times \sum_{n=-\infty}^{\infty} \psi_n e^{i\phi_n} + \text{conj.} \end{aligned}$$
  - $\phi_n = (k_0 + nk)x + (l_0 + nl)y$ contains wave perturbation (k_0, l_0)
- Sparse tridiagonal system
  - Orthogonality of basis functions + truncation
    ➔ Eigenvalue problem: $\mathbf{A}\vec{\psi} = \lambda\vec{\psi}$
  - Detect the most unstable eigenvalues



Instability contours

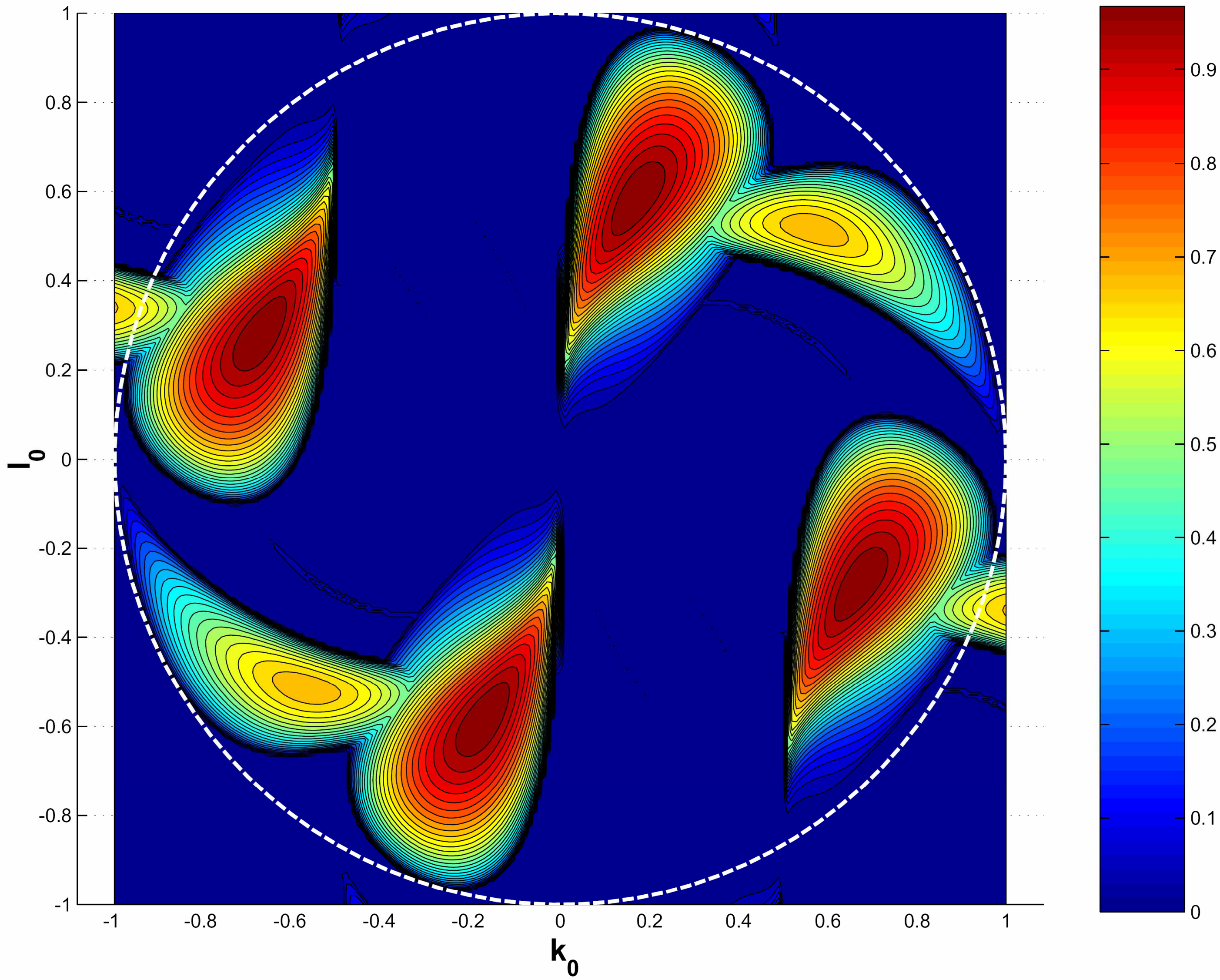


Fig. 4: Matlab *simulation of the stability for the disturbance function. The contour lines are imaginary parts of eigenvalues for different* (k_0,l_0)*, normalized w.r.t. to the highest imaginary part. The most unstable eigenvalue was found to have an imaginary part of -0.176.*

The angle between the wave vector of the basic solution and the x-axis is 120°, and the amplitude of this basic Rossby wave is 1.

Disturbed system

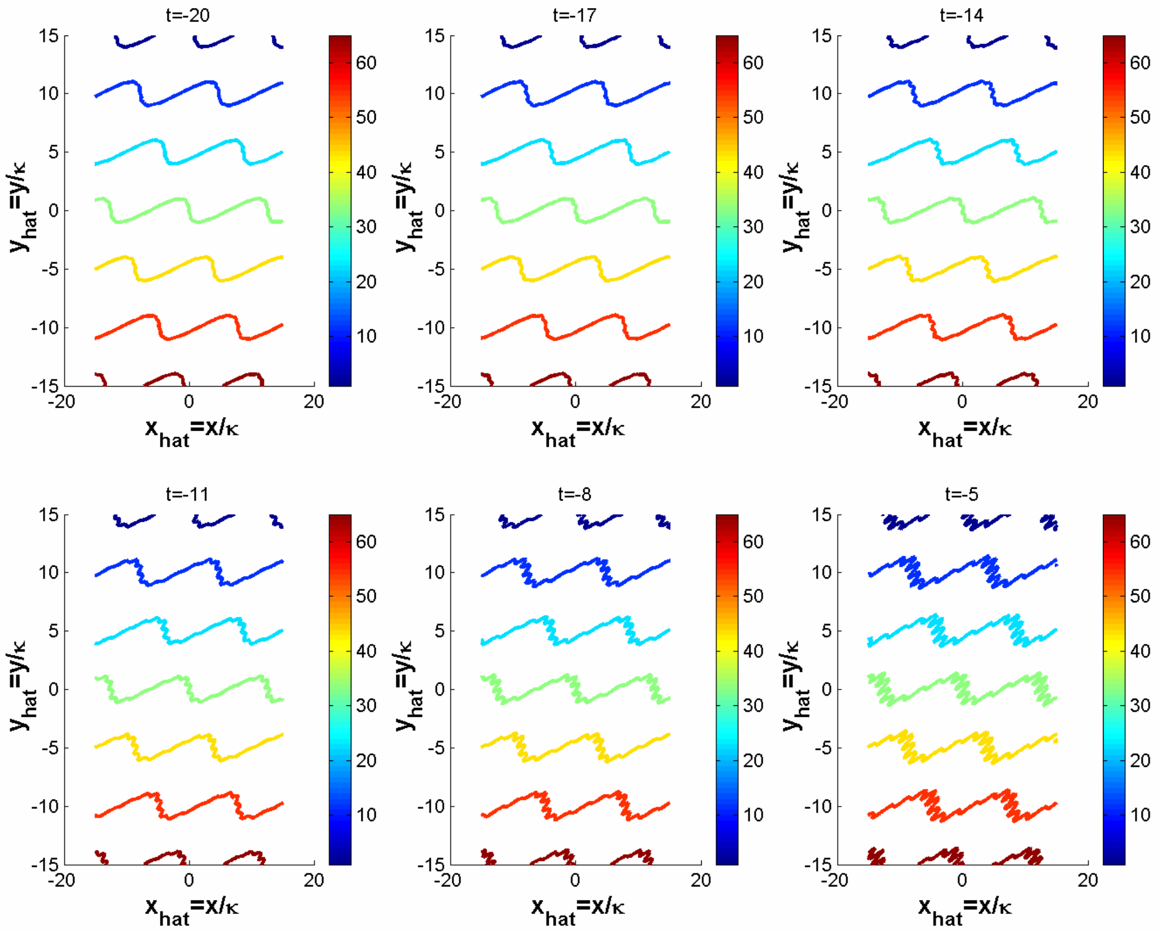


Fig. 5: Matlab *simulation of the onset of instability at the top of basic Rossby waves. (k_0,l_0) was chosen to be (0.2,0.6), corresponding to highly (red) instable region in Fig. 4 The streamfunctions are normalized.*

The angle between the wave vector of the basic solution and the x-axis is 120°, and the amplitude of this basic Rossby wave is 1.

However, those patterns are rarely seen on Earth, due to the dominant factors that spread the instabilities as soon as they develop.

Conclusion

- Conserved quantity: potential vorticity
- Rossby wave mechanism due to variation of Coriolis parameter with latitude
- Streamfunction to be interpreted as geopotential height or streamlines

References

- J.L. Anderson, *The instability of finite amplitude Rossby waves on the infinite beta-plane,* Geophys. Astrophys. Fluid Dynamics, Vol.63, pp.1 – 27, 1991
- B. Cushman-Roisin, *Introduction to Geophysical Fluid Dynamics*, Prentice Hall, New Jersey,1994
- G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press, 2006